Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

AIAA 81-4136

Transonic Flow in the Throat Region of Axisymmetric Nozzles

J. C. Dutton*
Texas A&M University, College Station, Texas
and
A. L. Addy†
University of Illinois at Urbana-Champaign Urbana, Ill.

Introduction

HALL[1] carried out a series expansion solution for the transonic flowfield in the throat region of both planar and axisymmetric nozzles. The transonic perturbation velocity components were expanded in appropriate series using $\epsilon = R_c^{-1}$ as the expansion parameter, where R_c is the wall radius of curvature at the nozzle throat nondimensionalized with the throat radius (for the axisymmetric case). The first three solution orders in the resulting formulation were obtained. However, this solution cannot be used to analyze nozzles for which the wall radius of curvature is on the order of, or less than, the throat radius since the series diverge in the limit as $\epsilon \to 1$. To remedy this deficiency, Kliegel and Levine[2] proposed a series expansion in $\epsilon = (R_c+1)^{-1}$ for axisymmetric nozzles. Since this parameter is less than unity even in the limit as $R_c \to 0$, the resulting series should have superior convergence properties in the limit of small R_c. Kliegel and Levine[2] contended that this series was the solution obtained when the one in toroidal coordinates, which are convenient for circular arc wall contours, was transformed back to cylindrical coordinates. However, in a later publication Levine and Coats[3] concluded that this contention was false and that the method "... is actually an empiricism [since] ... the proposed series do not satisfy the differential equations of motion in cylindrical coordinates." In reality, the series proposed in Ref. 2 is simply a clever transformation of Hall's[1] solution in $\epsilon = R_c^{-1}$ (as corrected by Kliegel and Levine[2]) to a series in $\epsilon = (R_c+1)^{-1}$ such that the two are equivalent in the limit of large radius of curvature, $R_c \to \infty$.

Since the expansion of Kliegel and Levine[2] is widely used, the purpose of this Note is to present the series solution for axisymmetric nozzles which satisfies the governing differential equations in cylindrical coordinates. The expansion variable employed is $\epsilon = (R_c+\eta)^{-1}$, where the value of the parameter η is arbitrary. The results of this solution are compared to two data sets for which the nozzle wall radius of curvature is relatively small.

Problem Formulation and Series Solution

The problem is formulated and solved in a manner similar to that originally used by Hall[1] with the exception of the introduction of the generalized expansion variable, $\epsilon = (R_c+\eta)^{-1}$. The dimensional R-Z cylindrical coordinate system is transformed to the normalized x-y system shown in the inset in Fig. 1 where all lengths have been nondimensionalized with respect to the throat radius. The governing equations for the velocity components can be taken as the irrotationality condition and the gas dynamic equation. Nondimensionalizing the velocity components with respect to the critical speed of sound and introducing the transonic perturbation velocity components $\tilde{u}$ and $\tilde{v}$, defined by:

$$u = 1 + \tilde{u} \tag{1}$$

$$v = \tilde{v} \tag{2}$$

the governing equations can be written as

$$\tilde{u}_y - \tilde{v}_x = 0 \tag{3}$$

$$\begin{aligned}&\left(-2\tilde{u} - \tilde{u}^2 - \frac{\gamma-1}{\gamma+1}\tilde{v}^2\right)\tilde{u}_x - \frac{4}{\gamma+1}(1+\tilde{u})\tilde{v}\tilde{u}_y \\ &+\left(\frac{2}{\gamma+1} - \tilde{v}^2 - 2\frac{\gamma-1}{\gamma+1}\tilde{u} - \frac{\gamma-1}{\gamma+1}\tilde{u}^2\right)\tilde{v}_y \\ &+\left(\frac{2}{\gamma+1} - 2\frac{\gamma-1}{\gamma+1}\tilde{u} - \frac{\gamma-1}{\gamma+1}\tilde{u}^2 - \frac{\gamma-1}{\gamma+1}\tilde{v}^2\right)\frac{\tilde{v}}{y} = 0\end{aligned} \tag{4}$$

where the subscripts denote partial differentiation and γ is the specific heat ratio. The boundary conditions are that both the axis of symmetry and the nozzle wall are streamlines,

$$\tilde{v}(x,0) = 0 \tag{5}$$

$$\tilde{v}(x,h(x)) = [1 + \tilde{u}(x,h(x))]\frac{dh}{dx} \tag{6}$$

The equation for the wall contour may be expanded in the form

$$h(x) = 1 + \left(\frac{1}{2}x^2\right)\epsilon + \left(\frac{1}{2}\eta x^2\right)\epsilon^2 + \left(\frac{1}{2}\eta^2 x^2 + \frac{\sigma}{8}x^4\right)\epsilon^3 + \ldots \tag{7}$$

where

$$\sigma = \begin{cases} 1 \text{ for circular arc contour} \\ 0 \text{ for parabolic arc contour} \\ -1 \text{ for rectangular hyperbolic arc contour} \end{cases}$$

Investigating the orders of magnitude of the dominant terms in Eqs. (3), (4), and (6) using Eq. (7) and the definition of the expansion variable, it is found that $\tilde{u} = O(\epsilon)$, $\tilde{v} = O(\epsilon^{3/2})$, and, for consistency, x must be taken as $O(\epsilon^{1/2})$ so that only the region near the throat plane, $x = 0$, can be considered. Introducing the $O(1)$, stretched, axial coordinate z defined by

$$z = \left[\frac{(\gamma+1)}{2}\epsilon\right]^{-1/2} x \tag{8}$$

Received Nov. 6, 1980.

*Assistant Professor, Dept. of Mechanical Engineering. Member AIAA.
†Professor and Associate Head, Dept. of Mechanical and Industrial Engineering. Associate Fellow AIAA.

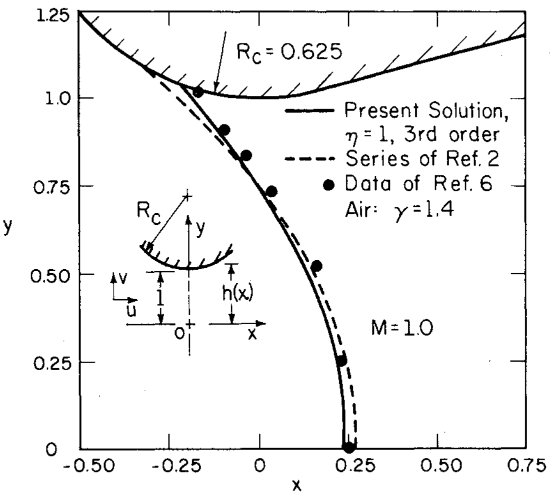

Fig. 1 Comparison of present solution, series of Ref. 2, and data of Ref. 6 for sonic line in an axisymmetric nozzle with $R_c = 0.625$.

the perturbation velocity components are expanded as

$$\bar{u}(z,y) = u_1(z,y)\epsilon + u_2(z,y)\epsilon^2 + u_3(z,y)\epsilon^3 + \ldots \tag{9}$$

$$\bar{v}(z,y) = \left[\frac{(\gamma+1)}{2}\epsilon\right]^{1/2}[v_1(z,y)\epsilon + v_2(z,y)\epsilon^2 + v_3(z,y)\epsilon^3 + \ldots] \tag{10}$$

The next step involves substitution of the coordinate defined by Eq. (8) and expansions (9) and (10) into the governing equations and boundary conditions, Eqs. (3-6), and gathering coefficients of like powers of ϵ. In boundary condition (6) the quantities $\bar{u}(x,h(x))$ and $\bar{v}(x,h(x))$ must first be expanded in Taylor series about $y=1$; thus, this condition is only approximately satisfied. The result of these operations is the formulations for the various orders in the expansion technique. The irrotationality condition becomes,

$$\frac{\partial u_n}{\partial y} - \frac{\partial v_n}{\partial z} = 0; \quad n = 1,2,3,\ldots \tag{11}$$

while the series of gas-dynamic equations is:

$$-u_1\frac{\partial u_1}{\partial z} + \frac{1}{2}\frac{\partial v_1}{\partial y} + \frac{1}{2}\frac{v_1}{y} = 0; \quad n=1 \tag{12}$$

$$-u_1\frac{\partial u_n}{\partial z} - u_n\frac{\partial u_1}{\partial z} + \frac{1}{2}\frac{\partial v_n}{\partial y} + \frac{1}{2}\frac{v_n}{y} = \phi_{n-1}(u_1,v_1,\ldots u_{n-1},v_{n-1}); \quad n=2,3,\ldots \tag{13}$$

where the functions ϕ_{n-1} are listed in Ref. 1. The boundary conditions are:

$$v_1(z,0) = 0 \quad v_1(z,1) = z \tag{14}$$

$$v_2(z,0) = 0 \quad v_2(z,1) = \eta z + z u_1(z,1) \tag{15}$$

$$v_3(z,0) = 0 \quad v_3(z,1) = \eta^2 z + \eta z u_1(z,1) + z u_2(z,1) - \left(\frac{\gamma+1}{4}\right) z^2 \frac{\partial v_1}{\partial y}\bigg|_{(z,1)} \tag{16}$$

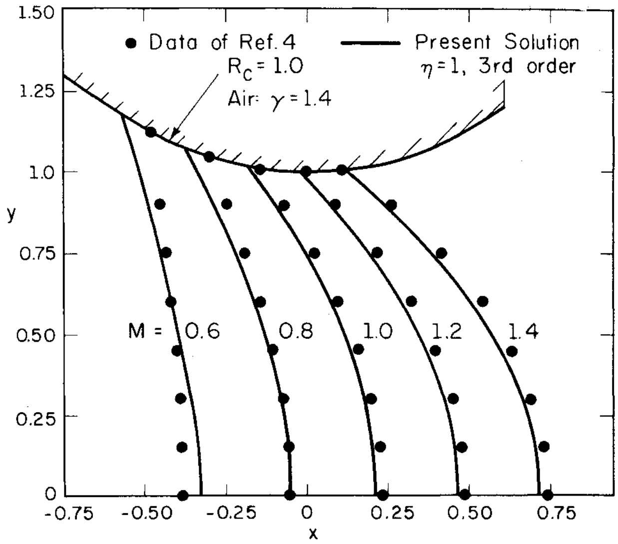

Fig. 2 Comparison of present solution and data of Ref. 4 for Mach number contours in an axisymmetric nozzle with $R_c = 1.0$.

The method used to solve these equations is based on assuming solution forms suggested by the boundary conditions. The resulting solutions for the first three orders are listed in the Appendix together with expansions for the dependent variables, including the velocity components, Mach number, discharge coefficient, etc. Through the first three orders, the solution is independent of the wall contour type, i.e., circular, parabolic, or hyperbolic arc, since the parameter σ introduced in Eq. (7) does not enter until the fourth order. This solution reduces to the corrected solution of Hall[1] for $\eta = 0$ but not to the series of Kliegel and Levine[2] for $\eta = 1$ since, as previously mentioned, their series do not satisfy the governing differential equations. In addition, the solution presented herein has been thoroughly checked[4] by an independent numerical back substitution into the governing equations and boundary conditions, Eqs. (11-16), using the computer program developed in Ref. 5 in which the derivatives were approximated by finite differences. A parametric study[4] has also been conducted to study the effect of the parameter η on the convergence properties of the solution, especially for nozzles with a small radius of curvature. For nozzles with large R_c, the solution is found to be essentially independent of η. For nozzles with small R_c, two competing effects appear: as η is increased from zero the convergence properties of the solution are significantly improved but the satisfaction of the exact wall boundary condition, Eq. (6), is compromised. For general use, third order, $\eta = 1$ solutions are recommended.

Comparison to Experiment

The results of the solution presented herein and the series proposed in Ref. 2 are compared in Fig. 1 to the sonic line data of Cuffel et al.[6] for an axisymmetric nozzle with $R_c = 0.625$. As can be seen in the figure, the present third order, $\eta = 1$ solution provides better agreement in the region near the nozzle wall where the results of Ref. 2 deviate significantly from the data.

Figure 2 presents a comparison between the present solution and data obtained in Ref. 4 for an axisymmetric nozzle with $R_c = 1.0$; the results are presented as contours of constant Mach number. In Ref. 4 a half-section model was constructed and mounted on a splitter plate which had 71 static pressure taps distributed through the throat region. The Mach number contours were interpolated from the resulting pressure measurements. Figure 2 demonstrates that the approximate solution given here may be used to predict accurately the throat flowfield for nozzles with a relatively small wall radius of curvature.

Appendix

Given the parameters γ, R_c, and η the expansion variable can be calculated as $\epsilon = (R_c + \eta)^{-1}$, and at the location (x,y) in the nozzle throat the perturbation velocity components and the discharge coefficient constants may be evaluated from

$$z = \left[\frac{(\gamma+1)}{2}\epsilon\right]^{-1/2} x \tag{A1}$$

$$u_1(z,y) = \tfrac{1}{2}y^2 - \tfrac{1}{4} + z \tag{A2}$$

$$v_1(z,y) = \tfrac{1}{4}y^3 - \tfrac{1}{4}y + yz \tag{A3}$$

$$u_2(z,y) = \frac{2\gamma+9}{24}y^4 - \frac{4\gamma+15-12\eta}{24}y^2 + \frac{10\gamma+57-72\eta}{288} + \left(y^2 + \frac{4\eta-5}{8}\right)z - \left(\frac{2\gamma-3}{6}\right)z^2 \tag{A4}$$

$$v_2(z,y) = \frac{\gamma+3}{9}y^5 - \frac{20\gamma+63-36\eta}{96}y^3 + \frac{28\gamma+93-108\eta}{288}y + \left(\frac{2\gamma+9}{6}y^3 - \frac{4\gamma+15-12\eta}{12}y\right)z + yz^2 \tag{A5}$$

$$\begin{aligned} u_3(z,y) = {} & \frac{556\gamma^2+1737\gamma+3069}{10{,}368}y^6 - \frac{388\gamma^2+(1161-384\eta)\gamma+(1881-1728\eta)}{2304}y^4 \\ & + \frac{304\gamma^2+(831-576\eta)\gamma+(1242-2160\eta+864\eta^2)}{1728}y^2 - \frac{2708\gamma^2+(7839-5760\eta)\gamma+(14{,}211-32{,}832\eta+20{,}736\eta^2)}{82{,}944} \\ & + \left[\frac{52\gamma^2+51\gamma+327}{384}y^4 - \frac{52\gamma^2+75\gamma+(279-288\eta)}{192}y^2 + \frac{92\gamma^2+180\gamma+(639-1080\eta+432\eta^2)}{1152}\right]z \\ & + \left[-\frac{7\gamma-3}{8}y^2 + \frac{(13-16\eta)\gamma-(27-24\eta)}{48}\right]z^2 + \left[\frac{4\gamma^2-57\gamma+27}{144}\right]z^3 \end{aligned} \tag{A6}$$

$$\begin{aligned} v_3(z,y) = {} & \frac{6836\gamma^2+23{,}031\gamma+30{,}627}{82{,}944}y^7 - \frac{3380\gamma^2+(11{,}391-3840\eta)\gamma+(15{,}291-11{,}520\eta)}{13{,}824}y^5 \\ & + \frac{3424\gamma^2+(11{,}271-7200\eta)\gamma+(15{,}228-22{,}680\eta+6480\eta^2)}{13{,}824}y^3 \\ & - \frac{7100\gamma^2+(22{,}311-20{,}160\eta)\gamma+(30{,}249-66{,}960\eta+38{,}880\eta^2)}{82{,}944}y + \left[\frac{556\gamma^2+1737\gamma+3069}{1728}y^5\right. \\ & \left. - \frac{388\gamma^2+(1161-384\eta)\gamma+(1881-1728\eta)}{576}y^3 + \frac{304\gamma^2+(831-576\eta)\gamma+(1242-2160\eta+864\eta^2)}{864}y\right]z \\ & + \left[\frac{52\gamma^2+51\gamma+327}{192}y^3 - \frac{52\gamma^2+75\gamma+(279-288\eta)}{192}y\right]z^2 + \left[-\frac{7\gamma-3}{12}y\right]z^3 \end{aligned} \tag{A7}$$

$$C_{D1} = 1/96 \tag{A8}$$

$$C_{D2} = -\frac{8\gamma+21-48\eta}{2304} \tag{A9}$$

$$C_{D3} = \frac{754\gamma^2+(1971-2880\eta)\gamma+(2007-7560\eta+8640\eta^2)}{276{,}480} \tag{A10}$$

With these quantities determined, the dependent variables of interest can be calculated. These include the velocity components u and v (nondimensionalized with respect to the critical speed of sound, a^*); the dimensionless speed ratio $M^* = V/a^*$; the angle of inclination of the velocity vector from the axis, θ; the Mach number, M; the static-to-stagnation pressure ratio, p/p_0; and the discharge coefficient, C_D. The following expansions are used:

$$u(z,y) = 1 + u_1\epsilon + u_2\epsilon^2 + u_3\epsilon^3 + \ldots \tag{A11}$$

$$v(z,y) = \left[\frac{(\gamma+1)}{2}\epsilon\right]^{1/2}[v_1\epsilon + v_2\epsilon^2 + v_3\epsilon^3 + \ldots] \tag{A12}$$

$$M^*(z,y) = 1 + u_1\epsilon + u_2\epsilon^2 + \left(u_3 + \frac{\gamma+1}{4}v_1^2\right)\epsilon^3 + \ldots \tag{A13}$$

$$\begin{aligned} \theta(z,y) = {} & \left[\frac{(\gamma+1)}{2}\epsilon\right]^{1/2}[v_1\epsilon + (v_2 - u_1v_1)\epsilon^2 \\ & + (v_3 - u_1v_2 - u_2v_1 + u_1^2v_1)\epsilon^3 + \ldots] \end{aligned} \tag{A14}$$

$$\begin{aligned} M(z,y) = 1 + \left(\frac{\gamma+1}{2}\right)\Big\{ & u_1\epsilon + \left[u_2 + \frac{3}{4}(\gamma-1)u_1^2\right]\epsilon^2 + \left[u_3 \right. \\ & \left. + \frac{\gamma+1}{4}v_1^2 + \frac{3}{2}(\gamma-1)u_1u_2 + \frac{(5\gamma^2-8\gamma+3)}{8}u_1^3\right]\epsilon^3 + \ldots\Big\} \end{aligned} \tag{A15}$$

$$\begin{aligned} \frac{p}{p_0}(z,y) = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \Big\{ 1 - \gamma\Big[& u_1\epsilon + u_2\epsilon^2 \\ & + \left(u_3 + \frac{\gamma+1}{4}v_1^2 - \frac{\gamma+1}{6}u_1^3\right)\epsilon^3 + \ldots\Big]\Big\} \end{aligned} \tag{A16}$$

$$C_D = 1 - (\gamma+1)\epsilon^2[C_{D1} + C_{D2}\epsilon + C_{D3}\epsilon^2 + \ldots] \tag{A17}$$

Acknowledgment

Support for this research was provided in part by the U.S. Army Research Office under Research Grant DAAG 29-76-G-0200 and Research Contract DAAG 29-79-C-0184.

References

[1] Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XV, Pt. 4, 1962, pp. 487-508.

[2] Kliegel, J. R. and Levine, J. N., "Transonic Flow in Small Throat Radius of Curvature Nozzles," *AIAA Journal*, Vol. 7, July 1969, pp. 1375-1378.

[3] Levine, J. N. and Coats, D. E., "Transonic Flow in a Converging-Diverging Nozzle," NASA CR111104, Sept. 1970.

[4] Dutton, J. C. and Addy, A. L., "A Theoretical and Experimental Investigation of Transonic Flow in the Throat Region of Annular Axisymmetric, Supersonic Nozzles," Dept. of Mechanical and Industrial Engineering, Univ. of Illinois at Urbana-Champaign, Rept. No. UILU-ENG-80-4001, Jan. 1980.

[5] Dutton, J. C. and Addy, A. L., "TRANNOZ: A Computer Program for Analysis of Transonic Throat Flow in Axisymmetric, Planar, and Annular Supersonic Nozzles," Dept. of Mechanical and Industrial Engineering, Univ. of Illinois at Urbana-Champaign, Rept. No. UILU-ENG-80-4005, April 1980.

[6] Cuffel, R. F., Back, L. H., and Massier, P. F., "Transonic Flowfield in a Supersonic Nozzle with Small Throat Radius of Curvature," *AIAA Journal*, Vol. 7, July 1969, pp. 1364-1366.

AIAA 81-4137

Pulsating Flows about Axisymmetric Concave Bodies

A. G. Panaras*
von Kármán Institute for Fluid Dynamics, Rhode St. Genese, Belgium

I. Introduction

SURFACE indentations in the nose region of axisymmetric blunt bodies (Fig. 1) in high-speed flow has been considered as a means of decreasing drag and heating in high-speed cruise or glide vehicles and increasing drag in planetary re-entry vehicles. If flow separation occurs, it is not always steady—it may be periodically unsteady. Two distinct modes of instability have been observed.

In the "pulsation" mode (Fig. 1a) the conical separation bubble formed on the concave part of the body periodically inflates and expands radially, taking a hemispherical shape. In the "oscillation" mode the conical foreshock, which envelops the separation bubble, and the accompanying shear layer oscillate laterally and their shape changes periodically from concave to convex (Fig. 1b). The pulsation mode was first observed by Mair[1] and the oscillation mode by Bogdonoff and Vas.[2] The terminology is due to Kabelitz.[3]

A comparison shows that the oscillating flow about a concave body is a subcase of the self-sustained oscillations of impinging free shear layers, reviewed by Rockwell and Naudascher.[4] Thus, the primary mechanism of the oscillation is related to the stability of the shear layer that envelops the conical separation bubble. It has been shown by the present author[5] that critical parameters for the occurrence of oscillation are the volume of the axisymmetric cavity and the geometry of the shoulder of the afterbody where the shear layer reattaches.

However, the mechanism of the pulsation is quite different. It will be shown in this Note that the pulsation is due to the effect which an annular supersonic jet, appearing at the shock intersection of the foreshock and aftershock, has on the separation bubble.

II. The Mechanism of the Pulsation Mode

Examination of the Schlieren photographs presented in Fig. 2 indicates that at the starting phase of each cycle of pulsation, the flow conditions prevailing resemble an impulsive flow in which the shock envelope retains a position corresponding to an inviscid flowfield. The persistence of the shock intersection during this initial phase, in which the separation bubble inflates (Figs. 2a-2c), plus the existing pressure imbalance behind the two shocks caused us to analyze the flow in the vicinity of the shock intersection and thus to discover the mechanism of this explosive instability.

Applying a quasisteady analysis at the shock intersection, we discovered that the shock formation is similar to one of the series studied by Edney[6] known to result in the production of a supersonic jet. This is schematically shown in Fig. 3, where the present unsteady case is compared with the Edney IV shock formation.

If the hypothesis of the existence of the supersonic jet is valid, the mechanism of the pulsation is as follows. For a certain fraction of the time of the cycle (which has been found experimentally[7] to be about one-tenth), the high-speed flow processed by the weak foreshock is channeled between the weak foreshock and the inflating separation bubble and is directed toward the body where it impinges as an annular supersonic jet. The separation bubble formed in the cavity of the body is thus inflated very rapidly, causing the distortion of the foreshock which forces the shock intersection away from the body and thus cutting off the source of the flow which causes the inflation. The high-pressure separated flow then expands and, since no disturbance remains to sustain the now strong foreshock, it collapses downstream and flows past the afterbody. Obviously, as the strong shock moves toward the afterbody, the forebody is exposed to the freestream, the weak shock reappears, and the initial conditions of the impulsive flow are re-established.

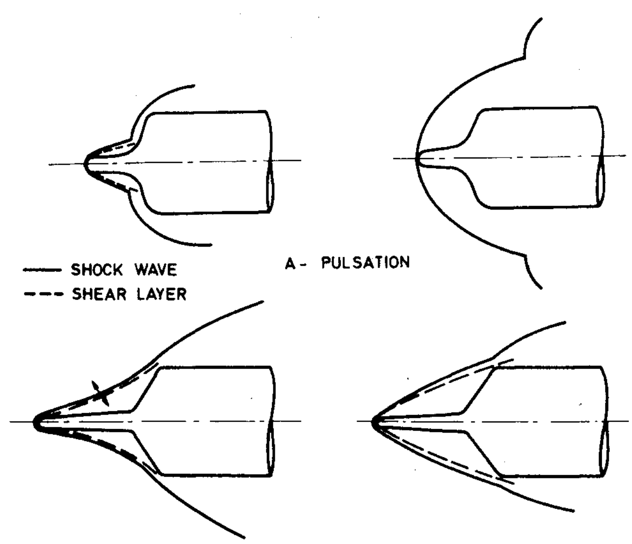

Fig. 1 Classification of instabilities.

Received May 8, 1980; revision received Jan. 29, 1981.

*Presently Special Scientist, Laboratory of Fluid Mechanics, University of Patras, Greece. Member of AIAA.